Dual Zariski Topology of Modules

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Throughout all rings will be associative with identity element and all modules will be unital right modules.

- A non-zero R-module M is called a **prime module** if $ann_R(M) = ann_R(K)$ for every non-zero submodule K of M.
- A proper submodule N of a module M is called a **prime submodule** of M if M/N is a prime module.
- If N is a prime submodule of a module M, then $p = ann_R(M/N)$ is a prime ideal of R and N is called p-prime submodule of M.

S. Yassemi introduced second submodules of modules over commutative rings as the dual notion of prime submodules.

- A non zero R-module M is called **a second module** if $\text{ann}_R(M) = \text{ann}_R(M/N)$ for every proper submodule N of M.
- A submodule N is called a second submodule if N is sumodule of M and second R–module, then .
- If N is a second submodule of a module M, then $ann_R(N) = P$ is a prime ideal of R and in this case N is called a P-**second submodule** of M.
- The set of all second submodules of a module M is called **the second spectrum** of M and denoted by $X^s = Spec^s(M)$.

- For a submodule N, $V^s(N) = \{S \in Spec^s(M) : ann_R(N) \subseteq ann_R(S)\}$ and $Z^s(M) = \{V^s(N) : N \leq M\}$.
- The **dual Zariski topology on** $\mathrm{Spec}^s(M)$ is the topology τ^s described by taking the set $Z^s(M)$ as the set of closed subsets of $\mathrm{Spec}^s(M)$

- Throughout the rest of the paper we assume that $Spec^s(M) \neq \emptyset$ for an R-module M.
- For a prime ideal p of R, $Spec^s_p(M)$ will denote the set of all p-second submodules of M.
- The map $\psi^s : Spec^s(M) \longrightarrow Spec(\overline{R})$ defined by $\psi^s(S) = \overline{ann_R(S)}$ is called **the natural map of** $Spec^s(M)$.
- M is said to be **secondful** if the natural map ψ^s is surjective.

Lemma

[2]The following statements are equivalent for an R-module M.

1. *The natural map* ψ^s :$Spec^s(M) \longrightarrow Spec(\overline{R})$ *is* **injective**.
2. *For any* $S_1, S_2 \in Spec^s(M)$*, if* $V^s(S_1) = V^s(S_2)$ *then* $S_1 = S_2$.
3. $|Spec^s_p(M)| \leq 1$ *for every* $p \in Spec(R)$.
4. $(Spec^s(M), \tau^s)$ *is a* T_0*-space.*

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4. $(Spec^s(M), \tau^s)$ *is a* T_0*-space.*

An R-module M is said to be X^s-**injective** if it satisfies one of the equivalent condition in Lemma.
It is well known that **every comultiplication module** is X^s-injective.

Let $\text{div}(M_R)$ denote the sum of all divisible submodules of the R-module M.

In the following Theorem, we characterize X^s-injective modules in terms of divisible submodules.

Theorem

Let M be an R-module. Then M is an X^s-injective R-module if and only if $S = \text{div}((0 :_M p)_{R/P})$ for every p-second submodule S of M.

Corollary

Let M be an X^s-injective R-module. Then:

- $Spec^s(M) = \{div((0 :_M p)_{R/p}) : p \in V(ann_R(M)),$
  $div((0 :_M p)) \neq 0\}.$
  $Min(M) = \{(0 :_M p) : p \in Max(R), (0 :_M p) \neq 0\}.$

Corollary

Let M be an X^s-injective R-module. Then:

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  $div((0:_M p)) \neq 0\}$.
  $Min(M) = \{(0:_M p) : p \in Max(R),\ (0:_M p) \neq 0\}$.
- *If M is secondful, then*
  $Spec^s(M) = \{div((0:_M p)_{R/p}) : p \in V(ann_R(M))\}$.
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  $Min(M) = \{(0 :_M p) : p \in V(ann_R(M)) \cap Max(R)\}$.
- *If R is a one-dimensional domain and M is a faithful secondful R-module, then*
  $Spec^s(M) = \{(0 :_M p) : p \in Max(R)\} \cup \{div(M_R)\}$.
  $Min(M) = \{(0 :_M p) : p \in Max(R)\}$.

In the following theorem, we deal with the maximal second submodules of an X^s-injective module.

Theorem

Let M be an X^s-injective R-module.

1. *Suppose that R is a one-dimensional integral domain and $S \in Spec^s_p(M)$ where p is a non-zero prime ideal of R. Then S is a maximal second submodule of M if and only if $S \not\subseteq div(M_R)$.*

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Let M be an X^s-injective R-module.

1. *Suppose that R is a one-dimensional integral domain and $S \in Spec_p^s(M)$ where p is a non-zero prime ideal of R. Then S is a maximal second submodule of M if and only if $S \not\subseteq div(M_R)$.*
2. *Let N be a submodule of M and $S \in Spec_p^s(M) \cap V^s(N)$ such that p is a minimal prime ideal of $ann_R(N)$. Then S is a maximal second submodule of N.*
   In particular, S is a maximal second submodule of $sec(N)$.

An R-module M is said to be **a weak comultiplication module** if M does not have any second submodule or for every second submodule S of M, $S = (0 :_M I)$, where I is an ideal of R.

Theorem

Then the following are true.

1. *If M is a weak comultiplication module, then M is X^s-injective.*

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Theorem

Then the following are true.

1. *If M is a weak comultiplication module, then M is X^s-injective.*
2. *If M is an injective R-module, then M is an X^s-injective R-module if and only if M is a weak comultiplication R-module.*

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2. *If M is an injective R-module, then M is an X^s-injective R-module if and only if M is a weak comultiplication R-module.*

The following example shows that the converse of Theorem-(1) is not true in general.

Example

Let p be a prime integer and M denote the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_p$. Then M is an X^s-injective R-module but M is not a weak comultiplication $\mathbb{Z}$-module. Indeed, the second submodule $\mathbb{Q} \oplus 0$ is not equal to $(0 :_M I)$ for some ideal I of $\mathbb{Z}$.

Theorem

Let R be a one-dimensional domain and M be an R-module. Then M is an X^s-injective R-module and M is either divisible or $div(M_R) = 0$ if and only if M is a weak comultiplication R-module.

Noetherian topological space

A topological space Y is said to be **Noetherian** if the open subsets of Y satisfy the ascending chain condition.
It is well-known that if Y is a Noetherian topological space, then every subspace of Y is quasi-compact.

Theorem

Let M be an R-module. Then $(Spec^s(M), \tau^s)$ is Noetherian in each of the following cases.
(1) R satisfies ascending chain condition on radical ideals.

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Theorem

Let M be an R-module. Then $(Spec^s(M), \tau^s)$ is Noetherian in each of the following cases.
(1) R satisfies ascending chain condition on radical ideals.
(2) M satisfies descending chain condition on the submodules of the form $(0 :_M I)$, where I is an ideal of R.

Corollary

Let M be an R-module. If $Spec(R)$ is a Noetherian topological space, then $(Spec^s(M), \tau^s)$ is a Noetherian space.

- A topological space X is called **irreducible** if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty.

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- A maximal irreducible subspace of X is called an **irreducible component**.

Theorem

Let R be a Dedekind domain and M be an R-module such that M is an X^s-injective R-module. Then we have the following.

1. $(Spec^s(M), \tau^s)$ *is a Noetherian topological space.*

Theorem

Let R be a Dedekind domain and M be an R-module such that M is an X^s-injective R-module. Then we have the following.

1. $(Spec^s(M), \tau^s)$ *is a Noetherian topological space.*
2. $(Spec^s(M), \tau^s)$ *is a T_1-space if and only if $\tau^s = \tau^{fc}$ where τ^{fc} is the finite complement topology.*

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1. $(Spec^s(M), \tau^s)$ *is a Noetherian topological space.*
2. $(Spec^s(M), \tau^s)$ *is a* T_1*-space if and only if* $\tau^s = \tau^{fc}$ *where* τ^{fc} *is the finite complement topology.*
3. *If* $Spec^s(M)$ *is finite, then* $(Spec^s(M), \tau^s)$ *is a* T_1*-space if and only if it is a* T_2*-space.*

Theorem

Let R be a Dedekind domain and M be an R-module such that M is an X^s-injective R-module. Then we have the following.

1. $(Spec^s(M), \tau^s)$ *is a Noetherian topological space.*
2. $(Spec^s(M), \tau^s)$ *is a T_1-space if and only if* $\tau^s = \tau^{fc}$ *where* τ^{fc} *is the finite complement topology.*
3. *If* $Spec^s(M)$ *is finite, then* $(Spec^s(M), \tau^s)$ *is a T_1-space if and only if it is a T_2-space.*
4. *If* $Spec^s(M)$ *is infinite, then* $(Spec^s(M), \tau^s)$ *is irreducible but not* T_2.

A spectral space

A topological space X is said to be **a spectral space** if X is homeomorphic to Spec(S), with the Zariski topology, for some commutative ring S.
Let Y be a closed subset of a topological space. An element $y \in Y$ is called **a generic point of** Y if $Y = cl(\{y\})$.
If the topological space is a T_0-space, then the generic point of an irreducible closed subset Y of a topological space is unique.

Spectral spaces were characterized by Hochster [6, p. 52, Proposition 4] as the topological spaces X which satisfy the following conditions:

1. X is a T_0-space;
2. X is compact and has a basis of compact open subsets;
3. The family of compact open subsets of X is closed under finite intersections;
4. Every irreducible closed subset of X has a generic point.

Theorem

Let R be a Dedekind domain and M an X^s-injective R-module. Then $(Spec^s(M), \tau^s)$ is a spectral space if and only if $Spec^s(M)$ is finite or $div(M_R) \neq 0$.

Theorem

Let M be an R-module. Then every irreducible closed subset of $(Spec^s(M), \tau^s)$ has a generic point in each of the following cases;
a)R is a zero-dimensional ring.

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Theorem

Let M be an R-module. Then every irreducible closed subset of $(Spec^s(M), \tau^s)$ has a generic point in each of the following cases;

a) R is a zero-dimensional ring.

b) R is a one-dimensional integral domain and M has at least one 0-second submodule.

Theorem

Let M be an R-module.

1. *Assume that R is a ring with Noetherian spectrum and M is an injective R-module. Then $(Spec^s(M), \tau^s)$ is a spectral space if and only if it is T_0-space.*

Theorem

Let M be an R-module.

1. *Assume that R is a ring with Noetherian spectrum and M is an injective R-module. Then $(Spec^s(M), \tau^s)$ is a spectral space if and only if it is T_0-space.*
2. *Assume that R is a one-dimensional integral domain, M has at least one 0-second submodule and $(Spec^s(M), \tau^s)$ is a Noetherian space. Then $(Spec^s(M), \tau^s)$ is a spectral space if and only if it is T_0-space.*

Theorem

Let M be an R-module.

1. *Assume that R is a ring with Noetherian spectrum and M is an injective R-module. Then $(Spec^s(M), \tau^s)$ is a spectral space if and only if it is T_0-space.*
2. *Assume that R is a one-dimensional integral domain, M has at least one 0-second submodule and $(Spec^s(M), \tau^s)$ is a Noetherian space. Then $(Spec^s(M), \tau^s)$ is a spectral space if and only if it is T_0-space.*
3. *If R is a one-dimensional integral domain with Noetherian spectrum and M is a divisible weak comultiplication R-module, then $(Spec^s(M), \tau^s)$ is a spectral space.*

Combinatorial Dimension

Let T be a topological space. We consider strictly decreasing (or strictly increasing) chain $Z_0, Z_1, ..., Z_r$ of length r of irreducible closed subsets Z_i of T.
The supremum of the lengths taken over all such chains is called the **combinatorial dimension** of T and denoted by $\dim T$. For the empty set $\emptyset$, the combinatorial dimension of $\emptyset$ is defined to be -1 [5, Definitions/Remarks 5.5].

Lemma

Let M be a secondful R-module and N be a submodule of M. Let Y be a non-empty subset of the closed set $V^s(N)$. Then
Y is an irreducible closed subset of $V^s(N)$ if and only if $Y = V^s(S)$ for some $S \in V^s(N)$.

Theorem

Let M be a secondful R-module, N be a submodule of M. Then:

1. *The mapping $\varrho : S \longmapsto V^s(S)$ is a surjection of $V^s(N)$ onto the set of irreducible closed subset of $V^s(N)$.*

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1. *The mapping $\varrho : S \longmapsto V^s(S)$ is a surjection of $V^s(N)$ onto the set of irreducible closed subset of $V^s(N)$.*
2. *The mapping $\varphi : V^s(S) \longmapsto \overline{ann_R(S)}$ is a bijection of the set of irreducible components of $V^s(N)$ onto the set of minimal prime ideals of $\overline{ann_R(N)}$ in $\overline{R}$.*

We recall two well-known facts of Noetherian spaces: (i) every subspace of a Noetherian space is Noetherian, and (ii) every Noetherian space has only finitely many irreducible components [3, p. 123, Proposition 8-(i)], [3, p. 124, Proposition 10]. By using these facts and Theorem, we obtain the following theorem.

Theorem

Let M be a secondful R-module. Then, every closed subset of $(Spec^s(M), \tau^s)$ has a finite number of irreducible components if and only if, for every $N \leq M$, the ideal $\overline{ann_R(N)}$ has a finite number of minimal prime ideals in $\overline{R}$.

Theorem

Let M be a secondful R-module. Then $(Spec^s(M), \tau^s)$ has a chain of irreducible closed subsets of length r if and only if $\overline{R}$ has a chain of prime ideals of length r.

Corollary

Let M be a secondful R-module. Then the combinatorial dimension of $(Spec^s(M), \tau^s)$ and the Krull dimension of $\overline{R}$ are all equal.

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   3. *Every irreducible closed subset of $(Spec^s(M), \tau^s)$ is an irreducible component of $(Spec^s(M), \tau^s)$.*
   4. *For every $p \in V(ann_R(M))$ and for every p-second submodule S of M, $Spec^s_p(M) = V^s(S)$.*

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2. *If one of the equivalent conditions in part (1) is satisfied and $(Spec^s(M), \tau^s)$ is a Noetherian space, then the set of irreducible components of $(Spec^s(M), \tau^s)$ is*

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2. *If one of the equivalent conditions in part (1) is satisfied and $(Spec^s(M), \tau^s)$ is a Noetherian space, then the set of irreducible components of $(Spec^s(M), \tau^s)$ is $\{V^s((0 :_M m_1)), ..., V^s((0 :_M m_k))\}$ for some $k \in \mathbb{Z}^+$, where the m_i, for $i = 1, ..., k$ are all the minimal prime ideals of $ann_R(M)$.*

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Thank you for your attentions.